UNIT 6 : POWER SERIES

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6.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- notion of power series
- radius of convergence interval of convergence of power series
- uniqueness of power series
- properties of power series.

6.2 INTRODUCTION

Power series is our important type of series in analysis. They can be used to define transcendental functions such as the exponential and trigonometric functions, and many other functions. In this unit we will study power series including radius of convergence of power series, uniqueness of power series, properties of power series, etc.

6.3 POWER SERIES

An infinite series of the form

$$a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ...\infty$$

where $a_0, a_1, a_2, a_3, ...; x_0$ are real numbers, is called a power series about the point c. c is called the centre of the power series.

Taking c = 0, we get the power series about 0, viz

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ...\infty$$

It can be denoted by $\sum_{n=0}^{\infty} a_n x^n$

Examples

1. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$

   i.e. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a power series

   We note that $\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{1}{n+1}\, x$

   $\therefore \left| \frac{u_{n+1}}{u_n} \right| \to 0 \text{ as } n \to \infty$ for all real x.

2. The geometric series $1 + x + x^2 + x^0 + ...\infty$

   i.e. $\sum_{n=0}^{\infty} x^n$ is a power series

   We know that the series converges for $-1 < x < 1$

3. The series $1 + x + 2!\, x^2 + 3!\, x^3 + ...\infty$

   i.e. $\sum_{n=0}^{\infty} n!\, x^n$ is a power series

   We note that $\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{n!} \cdot \frac{x^{n+1}}{x^n} = (n+1)\, x$

   $\therefore \left| \frac{u_{n+1}}{u_n} \right| \to \infty \text{ as } n \to \infty$, except when x = 0

   i.e. the series converges only for x = 0

We note that

(i) some power series converge for all reals x, which are called everywhere convergent power series;

(ii) some power series converge for some real x, and diverge for others;

(iii) some power series converge only for x = 0, which are called nowhere convergent power series.

Convergence of power series : The convergence of a power series can be examined by using d'Alembert's ratio test or Cauchy's root test. Ratio test or Root test can be combined with the absolute convergence theorem. They be applied to general series. These two tests are named as strengthened Ratio Test and strengthened Root Test.

Strengthened Ratio Test : If $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|$ exists and is equal to L, then the series $\sum u_n$ converges of L < 1, and diverges if L > 1.

Strengthened Root Test : If $\lim_{n \to \infty} |u_n|^{\frac{1}{n}}$ exists and is equal to L, then the series $\sum u_n$ converges if L < 1, and diverges if L > 1.

No conclusion can be drawn if L = 1. If L = ∞, the series diverges.



CHECK YOUR PROGRESS 1

Q 1: Examine the convergence of the following power series

(i) $x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots\ldots\ldots\infty$

(ii) $\sum_{n=1}^{\infty} \frac{1}{5^n}(x-3)^n$

$\sum_{n=1}^{\infty} \frac{x^4}{n!}$

6.4 RADIUS OF CONVERGENCE : INTERVAL OF CONVERGENCE

Theorem 6.1 If the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = x_0$ $(x_0 \neq 0)$ than it converges absolutely for all x with $|x| < |x_0|$.

Proof : Since the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = x_0 \ (x_0 \neq 0)$, therefore its terms converge to zero. Therefore, they are bounded and there exists $M \geq 0$ such that

$$\left| a_n x_0^n \right| \leq M, \quad \text{for n = 0, 1, 2, ...}$$

If $|x| < |x_0|$, then

$$\left| a_n x^n \right| = \left| a_n x_0^n \right| \left| \frac{x}{x_0} \right|^n$$

$$\leq M r^n, \quad r = \left| \frac{x}{x_0} \right| < 1$$

Since r < 1, $\sum M r^n$ is a convergent geometric series.

Therefore, by comparison test, the power series $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent.

Thus, if the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = x_0 \ (x_0 \neq 0)$, then it converges absolutely for all x with $|x| < |x_0|$.

Theorem 6.2 For any power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, one of the following three posibities holds good.

I. The series converges absolutely for every x.

II. The series converges for x = c, and diverges for other x.

III. There exists $0 \leq R \leq \infty$ such that the series converges absolutely for x with |x – c| < R, i.e. for x with c – R < x < c + R, and diverges for | x – c | > R.

The series may or may not converge for x = c – R or x = c + R.

Proof : We assume, without loss of generality, that c = 0. Then the series becomes

$$\sum_{n=1}^{\infty} a_n x^n$$

Let R = sup $\left\{|x| \geq 0 : \sum a_n x^n \text{ converges}\right\}$

If R = 0, then the series converges only for x = 0.

If R > 0, then the series converges absolutely for every real x with | x | < R, because it converges for some real number x_0 with | x | < | x_0 |

Furthermore the definition of R implies that the series diverges for every real x with | x | > R.

If R = ∞, then the series converges for all real x.

Definition :

If power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges for | x – c | < R, and diverges for | x – c | > R, then $0 \leq R \leq \infty$ is called the radius of convergence of the power series.

The set of all points where the power series converges in known as interval of convergence. It is onle of the following

$]c-R, c+R[$, $]c-R, c+R]$

$]c-R, c+R[$, $[c-R, c+R]$

Determination of radius of convergences, interval of convergence

We can computer the radius of convergence with the help of strengthened ratio that. Of course, it cannot be applied to every power series.

Theorem 6.3 Suppose that $a_n \neq 0$ for all sufficiently large n and the limit

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists or diverges to infinity. Then R is the radius of convergence of the power series.

$$\sum_{n=0}^{\infty} a_n (x-c)^n .$$

Proof : Let

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1} (x-c)^{n+1}}{a_n (x-c)^n} \right|$$

$$= |x-c| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then the power series converges if $0 \le r < 1$, i.e. if | x– c | < R, and diverges if $1 < r \le \infty$, i.e., if | x–c | > R.

The following theorem, due to the French mathematician Jacques Saloma Hadamard (1865 – 1963), can be applied for computing the radius of convergence of a general power series.

Theorem 6.4 The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n \text{ is given by}$$

$$R = \frac{1}{\lim\limits_{n \to \infty} \sup |a_n|^{\frac{1}{n}}}$$

where R = 0 if the lim sup diverges to ∞, and R = ∞ if the limit sup is 0.

Proof : Let

$$R = \lim_{n \to \infty} \sup \left| a_n (x-c)^n \right|^{\frac{1}{n}}$$

$$= |x-c| \lim_{n \to \infty} \sup |a_n|^{\frac{1}{n}}$$

Then the power series converges if $0 \le r < 1$, i.e., if | x – c | < R, and diverges if $1 < r \le \infty$, i.e., if | x – c | > R.

Thus, the result is proved.



CHECK YOUR PROGRESS 2

Q 1: Find the radius of convergence, and the interval of convergence of each of the following series

(i) $x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots\ldots\infty$

(ii) $\sum_{n=1}^{\infty} \frac{1}{5^n} (x-c)^n$

(iii) $\sum_{n=0}^{\infty} \frac{x^4}{n!}$

6.5 UNIQUENESS OF POWER SERIES

Now, we consider the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence R > 0. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

for all $x \in]\, c-R,\quad c+R\,[$

Then f is differentiable on] c – R, c + R [and for each $x \in]\, c-R,\quad c+R\,[$

Uniqueness of Power Series representation

If $f'(x) = \sum_{n=0}^{\infty} n\, a_n (x-c)^n$

and $f(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$, then $a_n = b_n$, n = 0, 1, 2, 3.....

We have

$$f(x) = a_0 + a_1(x-c) + a_2 (x-c)^2 + \ldots\ldots\ldots\infty \qquad (1)$$

Also

$$f(x) = b_0 + b_1(x-c) + b_2 (x-c)^2 + \ldots\ldots\ldots\infty \qquad (2)$$

From (1), $f(c) = a_0$

From (2), $f(c) = b_0$

i.e., $a_0 = f(c) = b_0$

Since f is differentiable on] c – R, c + R[, we have

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n (x-c)^{n-1}$$

$$= a_1 + 2a_2 (x-c) + 3a_3 (x-c)^2 + \ldots\ldots\ldots\infty \qquad (3)$$

Also

$$f'(x) = \sum_{n=1}^{\infty} n\, b_n (x-c)^{n-1}$$

$$= b_1 + 2b_2 (x-c) + 3b_3 (x-c)^2 + \ldots\ldots\ldots\infty \qquad (4)$$

From (3) and (4), we get

$$a_1 = f'(c) = b_1$$

We note that f' is also differentiable on] c – R, c + R [

Now $f''(x) = \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,(x-c)^{n-2}$

$$= 2a_2 + 3.\,2\,a_3(x-c) + ...\infty \quad (5)$$

Also

$$f''(x) = \sum_{n=2}^{\infty} n\,(n-1)\,b_n\,(x-c)^{n-1}$$

$$= b\,a_2 + 3.\,2\,b_3(x-c) + ...\infty \quad (6)$$

From (5) and (6), we get

$$2a_2 = f''(c) = 2b_2$$

$$\therefore a_2 = \frac{f''(c)}{2} = b_2$$

Now f has derivatives of all orders on] c – R, c + R [with

$$f'''(x) = \sum_{n=3}^{\infty} n\,(n-1)\,(n-2)\,a_n\,(x-c)^{n-3}$$

and

$$f'''(x) = \sum_{n=k}^{\infty} n\,(n-D)\,(n-2)\,(n-3)\,.....(n-k+1)\,a_n\,(x-c)^{n-k}$$

$$\therefore f'''(c) = 3!\,a_3 \Rightarrow a_3 = \frac{f'''(c)}{3!}$$

and $f^k(c) = k!\,a_k \Rightarrow a_k = \frac{f^{(k)}(c)}{k!}$

Similarly, we will get

$$f'''(x) = \sum_{n=3}^{\infty} n\,(n-1)\,(n-2)\,b_n\,(x-c)^{n-3}$$

and $p^{(k)}_{(x)} = \sum_{n=k}^{\infty} n\,(n-1)\,(n-2)\,(n-3)\,.....(n-k+1)\,b_n\,(x-c)^{n-k}$

$$\therefore f'''(c) = 3!\,b_k \Rightarrow b_3 = \frac{f'''(c)}{3!}$$

and $f^k(c) = k!\,b_k \Rightarrow b_k = \frac{f^{(k)}(c)}{k!}$

Thus, $\therefore a_k = \frac{f^{(k)}(x)}{k!} = b_k$

$\therefore$ $a_n = b_n$, for n = 0, 1, 2, 3 .....

Note : f may have different representations at different centre points.

Do You Know?

Some important power series with their interval of convergence are given below.

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\infty; \quad -\infty < x < \infty$

2. $\log_e |(1+x)| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty; \quad -1 < x \le 1$

3. $\frac{1}{2}\log_e \left|\frac{(1+x)}{1-x}\right| = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots\infty; \quad -1 < x < 1$

4. $\text{Sin}\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\infty; \quad -\infty < x < \infty$

5. $\text{Cos}\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\infty; \quad -\infty < x < \infty$

6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7!} + \ldots\ldots\infty; \quad -1 < x \le 1$

7. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\infty$

   (i) If n is zero or a positive integer, the series or a positive.

   (ii) If n > 0, but is not an integer, the series converges for $-1 \le x \le 1$

   (iii) If –1 < n < 0, the series converges for $-1 < x \le 1$

   (iv) If $n \le 1$, the series converges for $-1 < x < 1$

6.6 PROPERTIES OF POWER SERIES

Let us consider the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ and $\sum_{n=0}^{\infty} b_n (x-c)^n$

with radius of convergence $R_1(>0)$, $R_2(>0)$ respectively.

Let f(x), and g(x) be the functions defined by the two power series respectively, defined for all $x \in I$, where,

$I =]\, c - R, c+R\, [$,

Where $R = \min \{R_1, R_2\}$.

We note that both the power series converge for all $x \in I$,

We state the following properties.

I. Equality of power series

The two power series defined by f(x) and g(x) are equal for all $x \in I$ if and only if

$a_n = b_n$ for all n = 0, 1, 2, ...

In particular, if $\sum_{n=0}^{\infty} a_n (x-c)^n = 0$ for all $x \in I$, then

$a_n = 0$ for all n = 0, 1, 2, ...

II. Term by term addition

For all $x \in I$, we have

$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-c)^n$

III. Multiplication of power series

We define

$d_0 = a_0\, b_0$, and inductivdy $d_n = \sum_{j=1}^{n} a_{n-j} b_j$

Then for all $x \in I$, the product of f(x) and g(x) is defined by

$f(x)\, g(x) = \sum_{n=0}^{\infty} d_n (x-c)^n$

We note that for any $n \geq 0$, the coefficient of x^n in

$\left(\sum_{j=0}^{\infty} a_j (x-c)^j \right) \left(\sum_{k=0}^{\infty} b_k (x-c)^k \right)$ is $d_n = \sum_{j=1}^{n} a_{n-j} b_j$

IV. Term-by-term differentiation

The term-by-term differentiation of the power series defined by f(x) is given by $\sum_{n=1}^{\infty} x a_n (x-c)^{n-1}$.

We note that the radius of convergence of this new series is also R_1.

V. Term-by-term integration

The term-by-term integration of the power series defined by f(x) is given by

$$C = \sum_{n=0}^{\infty} a_n \frac{(x+c)^{n+1}}{n+1}$$

We note that the radius of convergence of this new series is also R_1.

Example : Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with interval of convergence J–1, I [, and $\sum_{n=0}^{\infty} b_n x^n$ be a power series with interval of convergence]–2, 2[. Find the interval of convergence, and the radius interval of convergence, and the radius of convergence of the power series $\sum_{n=0}^{\infty} (a_n + b_n)^{x^n}$.

Solution : We have]–1, 1[$\cap$]–2, 2[=]–1, 1[

$\therefore$ the interval of convergence of the power series $\sum_{n=0}^{\infty} (a_n + b_n)^{x^n}$ is]–I, I[

radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 1

radius of convergence of $\sum_{n=0}^{\infty} b_n x^n$ is 2.

$\therefore$ radius of convergence of $\sum_{n=0}^{\infty} (a_n + b_n)^{x^n}$ is min {1, 2}, i.e., I.

Example : Using the power series representation $f(x) = \frac{1}{1-x} = 1+x+x^2+x^3+ \ldots \infty$, –1< x <1, find a power series representation for $g(x) = \frac{1}{(1-x)^2}$, –1 < x < 1

Hence evaluate the sum of the series $\sum_{n=0}^{\infty} \frac{n+1}{5^n}$.

Solution : We have

$$f(x) = \frac{1}{1-x}$$

$= 1 + x + x^2 + x^3 + \ldots \infty$

$= \sum_{n=0}^{\infty} x^4$, –1 < x < 1.

We have $g(x) = \frac{1}{(1-x)^2}$

$= f'(x)$

$= \frac{d}{dx}\left(\frac{1}{1-x}\right)$

$= \frac{d}{dx}(1 + x + x^2 + x^3 + \ldots \infty)$

$= 0 + 1 + 2x + 3x^2 + \ldots \infty$

$= \sum_{n=0}^{\infty} (n+1)x^n, -1 < x < 1$

Second Part

We have $\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$

Putting $x = \frac{1}{5}, \frac{1}{5} \in]-1, 1[$, we get

$\sum_{n=0}^{\infty} (n+1)\left(\frac{1}{5}\right)^n = \frac{1}{\left(1-\frac{1}{5}\right)^2}$ i.e. $\sum_{n=0}^{\infty} \frac{n+1}{5^n} = \frac{25}{16}$



6.7 LET US SUM UP

- An infinite series of the form

  $a_0 + a_1(x-c) + a_x(x-c)^2 + a_3(x-c)^3 \ldots\ldots\ldots\infty$ where

  $a_0, a_1, a_2, a_3 \ldots\ldots\ldots$; c are real numbers, is called a power series about the point c. C is called the centre of the power series.

  Taking c = 0, we get the power series about 0, viz,

  $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\ldots\infty$

  It can be denoted by $\sum_{n=0}^{\infty} a_n x^n$

- If the power series

  $\sum_{n=0}^{\infty} a_n (x-c)^n$

  converges for $|x-c| < R$, and diverges for $|x-c| > R$, than $0 \le R \le \infty$

is called the radius of convergence of the power series.

The set of all points where the power series converges is known as interval of convergence. It is one of the following

$]c-R,\ c+R[$, $]c-R,\ c+R]$, $[c-R,\ c+R[$, $[c-R,\ c+R]$

- Let us assume that the power series $\sum_{n=0}^{\infty} a_n\ (x-c)^n$ has radius of convergence R > 0.

Let f(x) = $\sum_{n=0}^{\infty} a_n\ (x-c)^n$, for all $x \in\ ]c-R,\ c+R[$

Then f is differentiable on] c – R, c + R [and for each $x \in\ ]c-R,\ c+R[$,

$$f'(x) = \sum_{n=1}^{\infty} n\ a_n\ (x-c)^{n-1}$$



6.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q1 (i) Here $u_n = \frac{x^n}{n^x}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} = \frac{n^2}{(n+1)^2}$$

$$\therefore \left|\frac{u_{n+1}}{u_n}\right| = |x| \left|\left(\frac{n}{(n+1)}\right)^2\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} |x| \left|\left(\frac{1}{(1+\frac{1}{n})}\right)^2\right|$$

$$= |x|$$

$\therefore$ The series converges for | x | < 1, and diverges for | x | > 1.

When x = 1, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

clearly, it is convergent (p-series, p = 2) when x = –1, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, i.e.

$$-\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+.....\infty$$

The sequences $\left\{1, \frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, ...\right\}$ is a monotonic decreasing sequence, and

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

$\therefore$ The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent.

Thus, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is convergent for $-1 \le x \le 1$, and diverges for other real values of x.

(ii) Here $u_n = \frac{1}{5^n}(x-3)^n$

$$\therefore |u_n|^{\frac{1}{n}} = \left|\frac{x-3}{5}\right|$$

$$\therefore \lim_{n\to\infty} |u_n|^{\frac{1}{n}} = \left|\frac{x-3}{5}\right|$$

$\therefore$ The series converges for $\left|\frac{x-3}{5}\right| < 1$, and diverges for $\left|\frac{x-3}{5}\right| > 1$

Now $\left|\frac{x-3}{5}\right| < 1$

$$\Rightarrow -1 < \left|\frac{x-3}{5}\right| < 1$$

$$\Rightarrow -5 < x-3 < 5$$

$$\Rightarrow -2 < x < 8$$

When $x = -2$, $u_n = \frac{1}{5^n}(-2-3)^n = (-1)^2$

$\therefore$ When x = –2, the series is $\sum u_n = \sum (-1)^n$, which is divergent.

Again , when x = 8, $u_n = \frac{1}{5^n}(8-3)^n = 1$

$\therefore$ when x = 8, the series is $\sum u_n = \sum 1$, which is divergent.

$\therefore$ The power series $\sum_{n=1}^{\infty} \frac{1}{5^n}(x-3)^n$ is convergent for –2 < x < 8,

and divergent for other real values of x.

Note : Here the region of convergence is] –2, 8 [, and its middle point 3 is the centre of the power series.

(iii) Here $u_n = \frac{x^n}{n!}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)!}.\frac{n!}{x} = \frac{n\,i}{(n+1)\,i}x = \frac{1}{(n+1)}x$$

$$\therefore \left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{(n+1)}.\ |x|$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{1}{(n+1)}.\ |x| = 0$$

The power series is convergent for all real x.

Check Your Progress 2

Ans to Q1 (i) Here $a_n = \frac{1}{n^2}, \quad c = 0$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}}\right|$$

$\therefore$ c – R = 0 – 1 = –1, c + R = 1

When x = 1, the series is convergent

when x = –1, the series is also convergent

$\therefore$ radius of convergence = 1,

interval of convergence is [–1, 1]

(ii) Here $a_n = \frac{1}{5^n}, \quad c = 3$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{5^n}}{\frac{1}{5^{n+1}}}\right| = \lim_{n\to\infty} 5 = 5$$

$\therefore$ c – R = 3 – 5 = –2, c + R = 8

when x = –2, the series is divergent

when x = 8, the series is also divergent

$\therefore$ radius of convergence = 5

interval of convergence in] –2, 8 [

(iii) Here $a_n = \frac{1}{n!}$, $c = 0$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty}(n+1) = \infty$$

$\therefore$ radius of convergence = ∞

interval of convergence is $]-\infty, \infty[$



6.9 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin
2) Methods of real Analysis by Richand R. Goldbery
3) Mathematical Analysis by S. C. Malik and Savita Arora
4) A Course of Mathematical Analysis by shanti Narayan.



6.10 MODEL QUESTIONS

Q 1: Find the radius of convergence, and the interval of convergence of each of the following series

(i) $\sum_{n=0}^{\infty} x^n$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n} x^n$

(iii) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

(iv) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

(v) $\sum_{n=0}^{\infty} n!\ x^4$

(vi) $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + ...\infty$

(vii) $1 + 3x + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} + ...\infty$

Q 2: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R, (R > 0). Construct a power series $\sum_{n=0}^{\infty} b_n x^n$, other than $\sum_{n=0}^{\infty} x^n$, such that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n\ b_n\ x^n$ is also R.

[We take $b_n = \frac{1}{n+1}$]

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